On $(K_q;k)$-Stable Graphs

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On (K_q;k)-Stable Graphs

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Introduction

- Definitions
- General Bounds
- Complete Graphs, K_q
- Future Work
- Conclusion

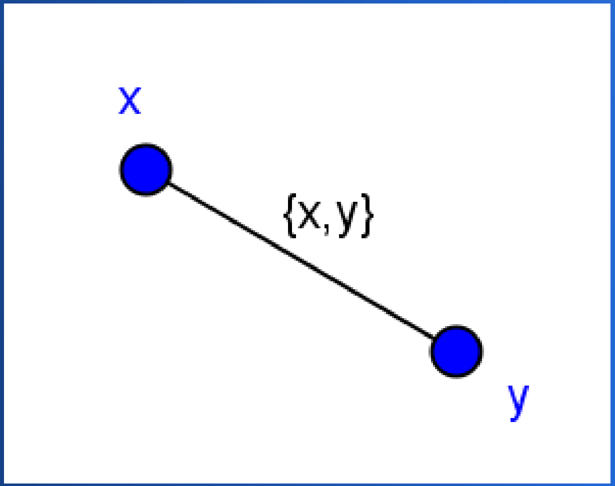
Definitions

- What is a graph?
  - Order of a graph.
  - Size of a graph.
- Graph Isomorphism.
- Cycles, C_n
- Complete Graphs, K_n
- Vertex Stable Graph.
  - Minimum vertex stable graph.
  - Size of a minimum vertex stable graph.

What is a graph?

A (simple) graph G consists of a *vertex* set V(G) and an *edge* set E(G) where E(G) is a set of 2-element subsets of V(G). When $\{x, y\} \in$ E(G) we write $x \sim y$. A vertex that does not belong to any edge is called an isolated vertex.

Let *x*,*y* be vertices of G, then if *x* and *y* share an edge we denote that edge as $\{x, y\}$.



What is a graph?

- Order of a Graph: Let G be a graph, then the *order* of G is defined as the number of vertices it has, denoted:

$$|G| := |V(G)|.$$

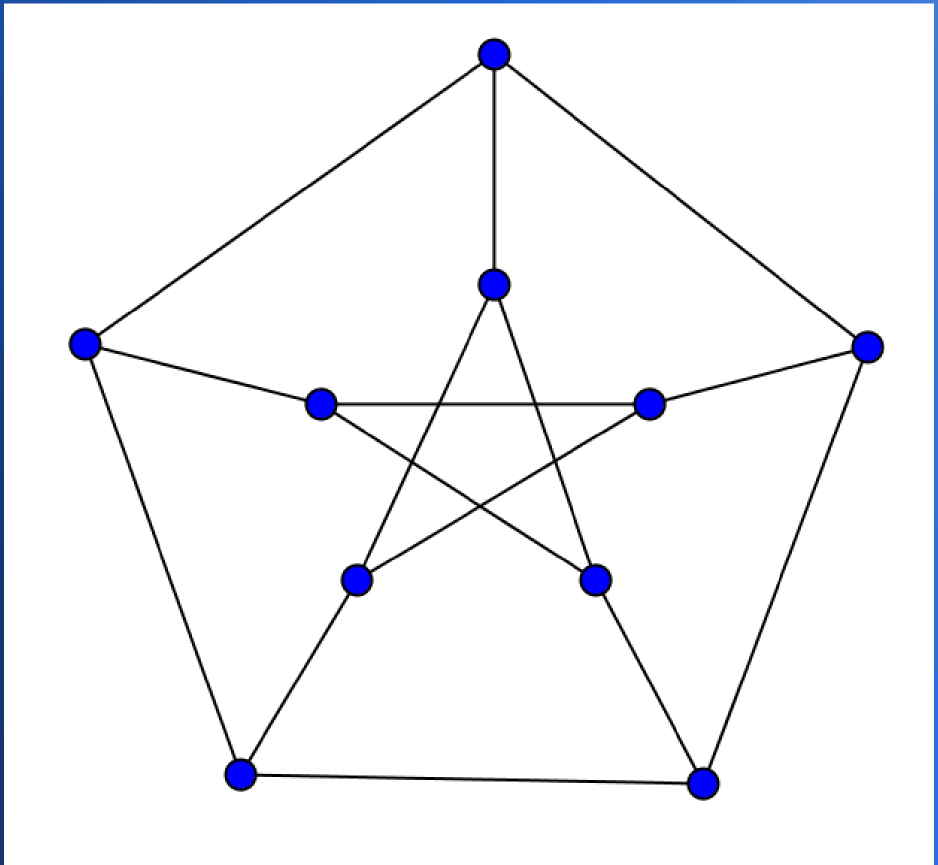
- Size of a Graph: Let G be a graph, then the *size* of G is defined as the number of edges it has, denoted:

$$\|G\| := |E(G)|.$$

Example

Call the following graph G.

- What is the Order of this graph?
- What is the size of the graph?



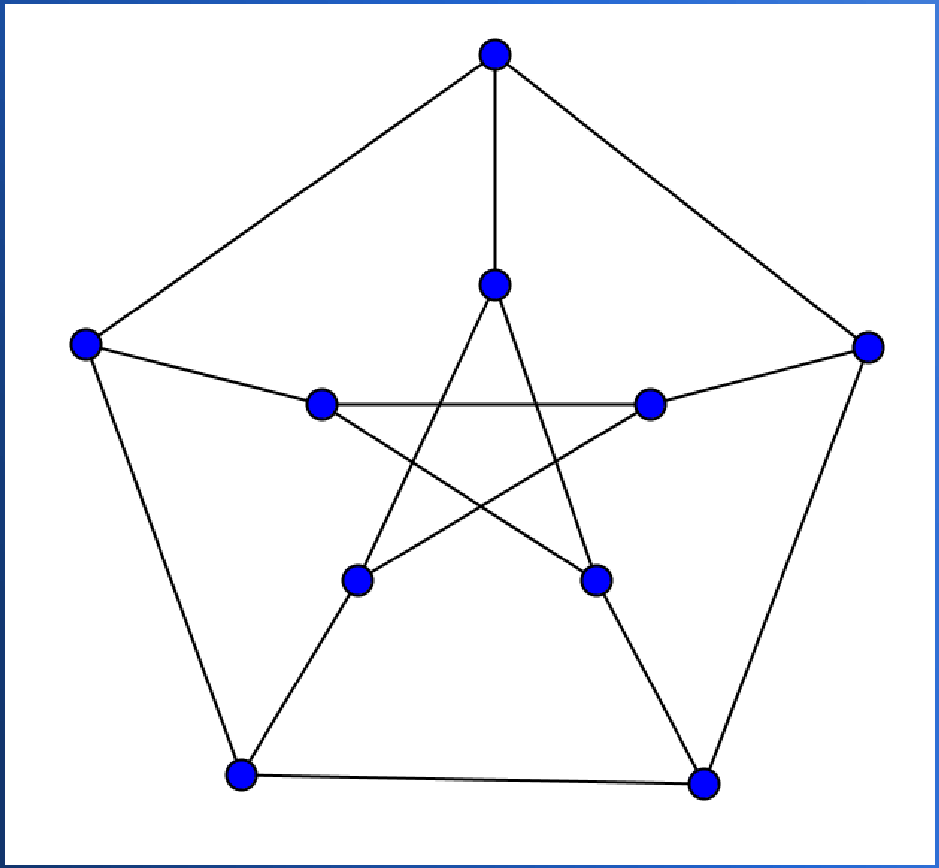
Example

- What is the Order of this graph?

$$|G| = 10$$

- What is the size of the graph?

$$\|G\| = 15$$



Graph Isomorphism

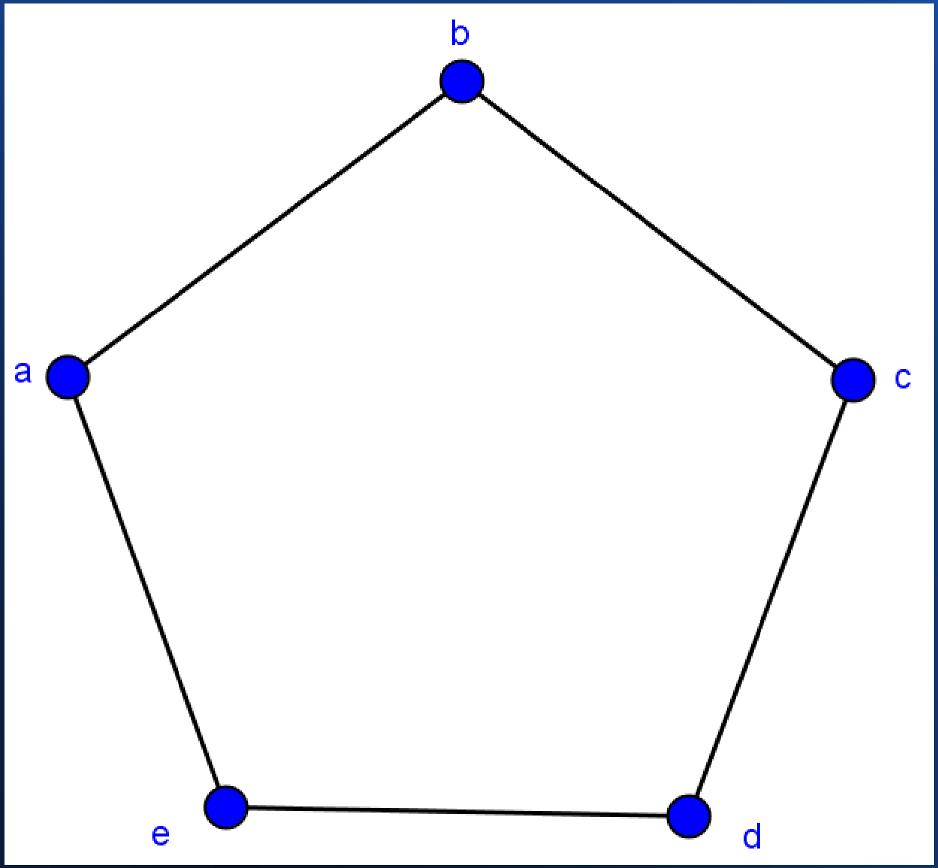
Two graphs G and H are *isomorphic* if there is a bijection

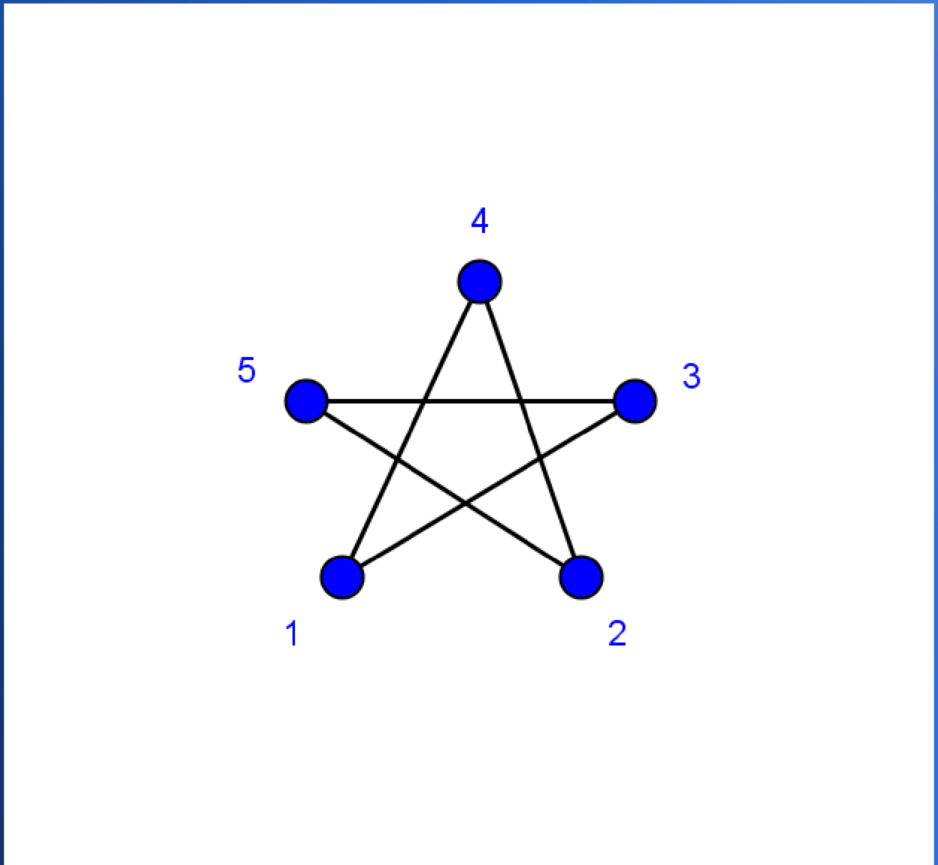
$$f : V(G) \to V(H)$$

such that

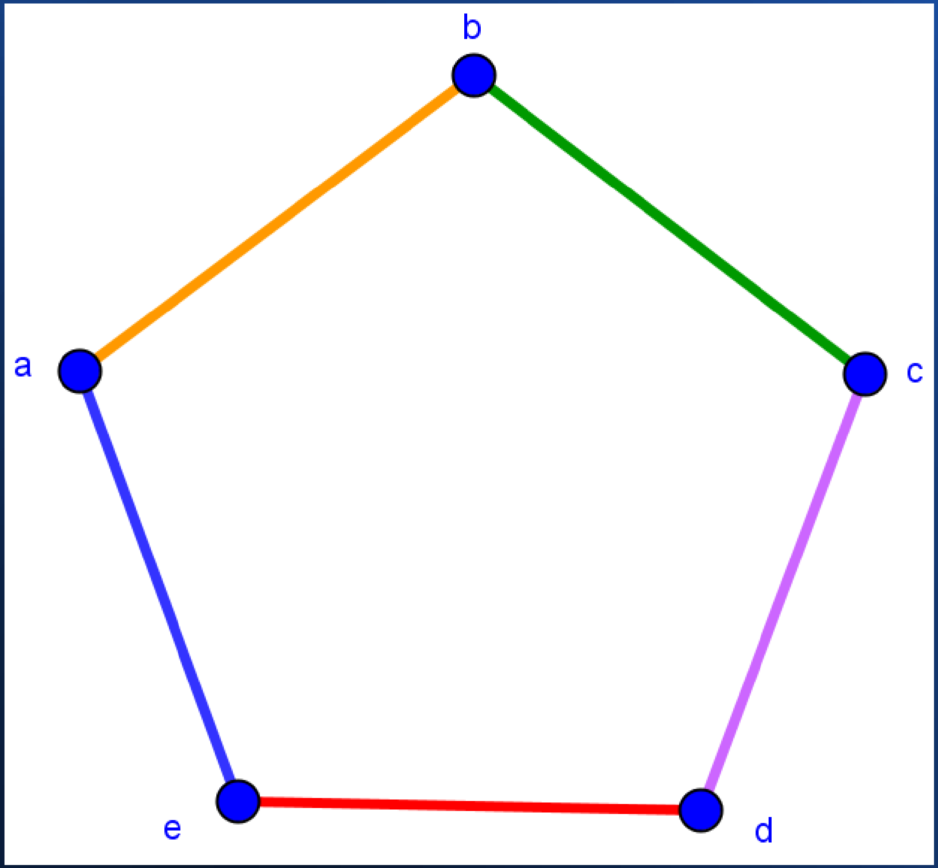
$$u \sim v \quad \mathit{iff} \quad f(u) \sim f(v).$$

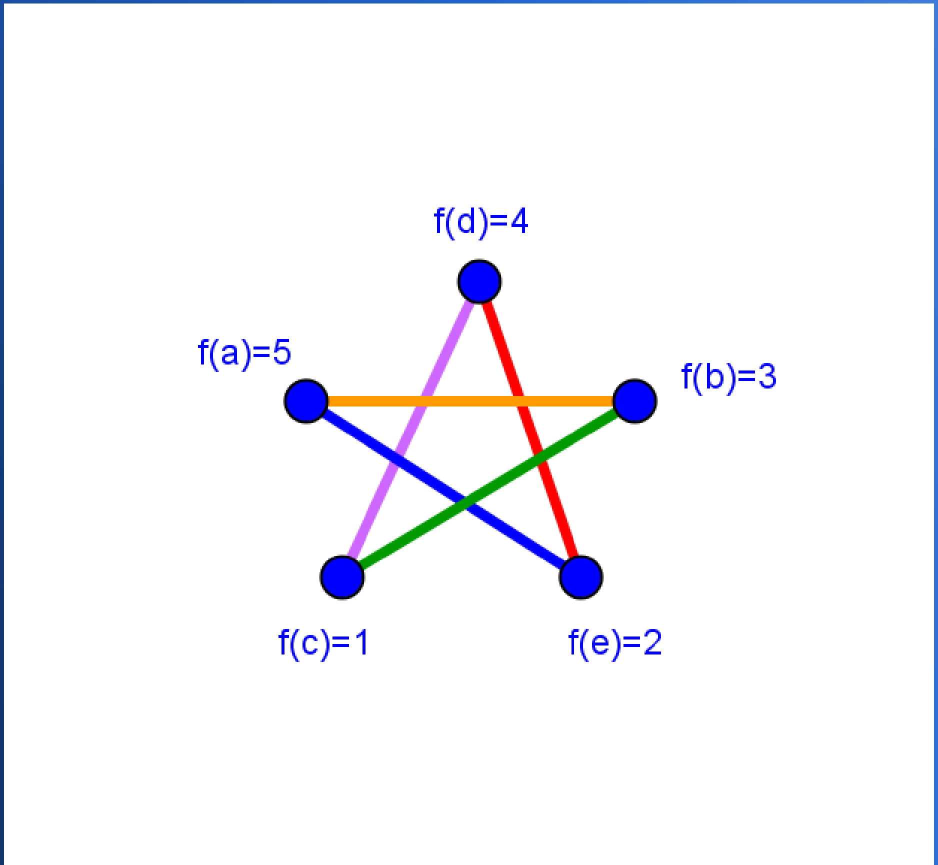
Example





Example

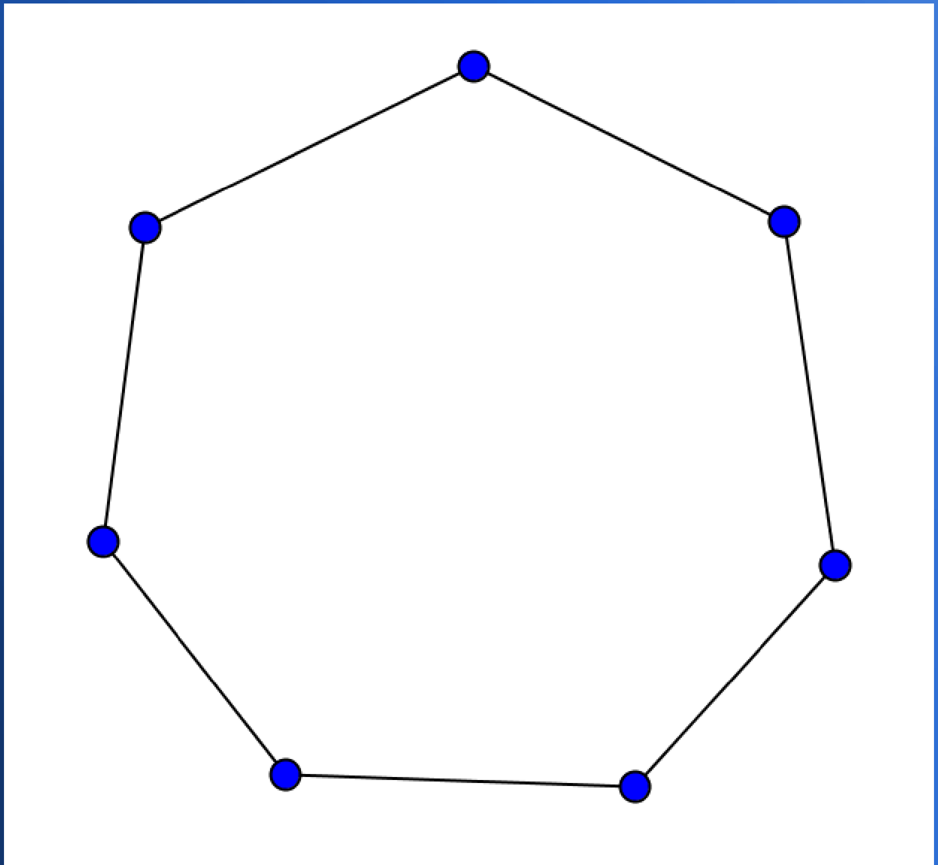




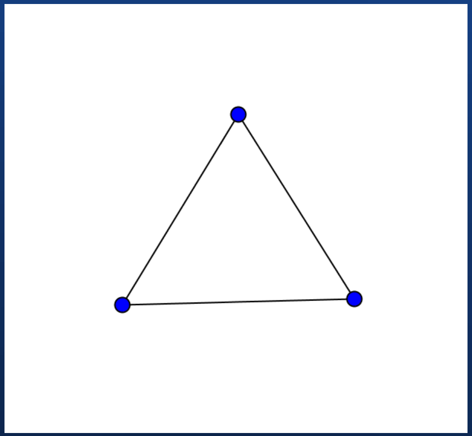
Cycle graphs, C_n

A cycle, more commonly called a closed walk, consists of a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph.

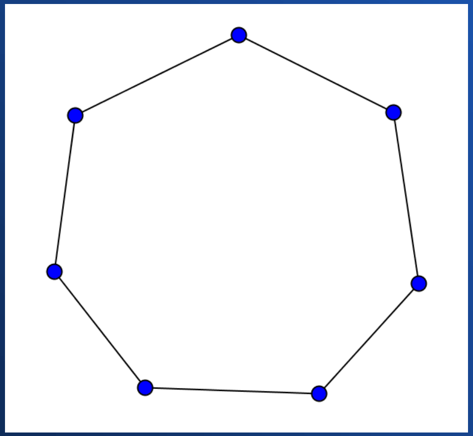
Example: C_7



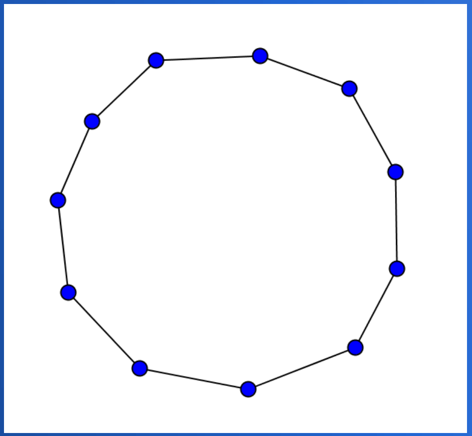
Cycle graphs, C_n



C_3



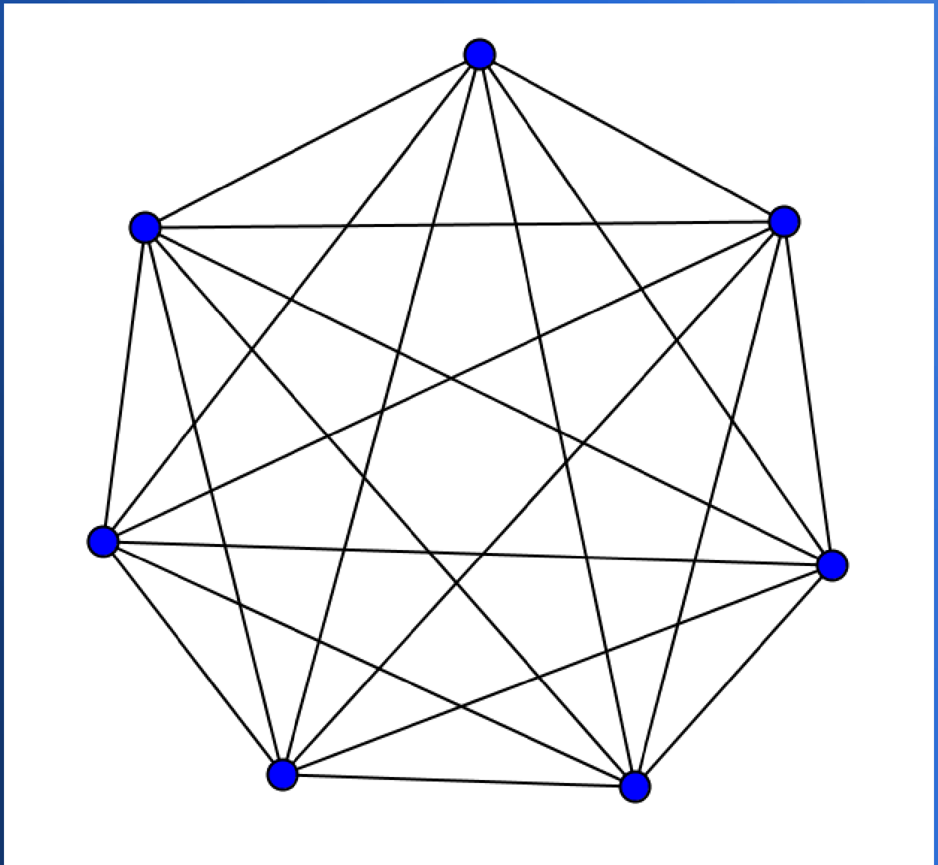
C_7



C_{11}

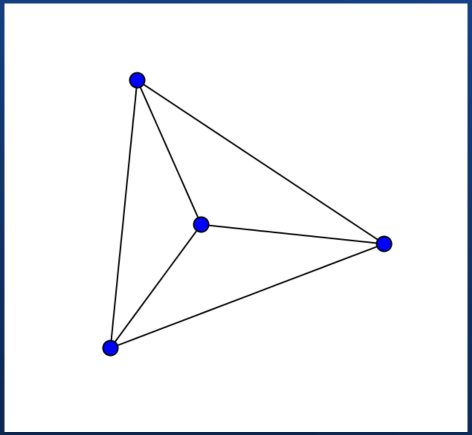
Complete Graphs, K_q

A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

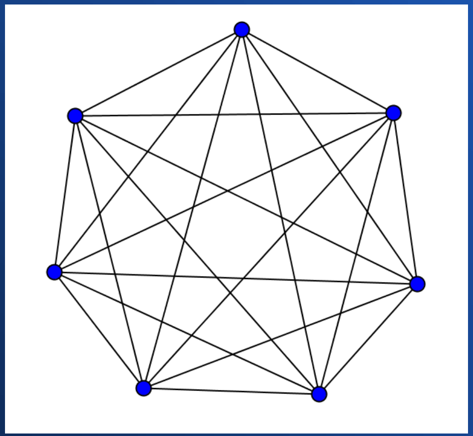


Example: K_7

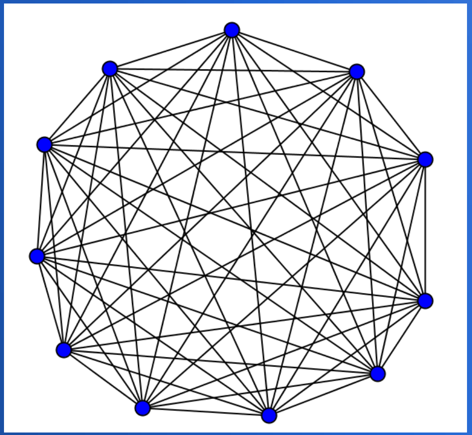
Complete Graphs, K_q



K_4



K_7



K_{11}

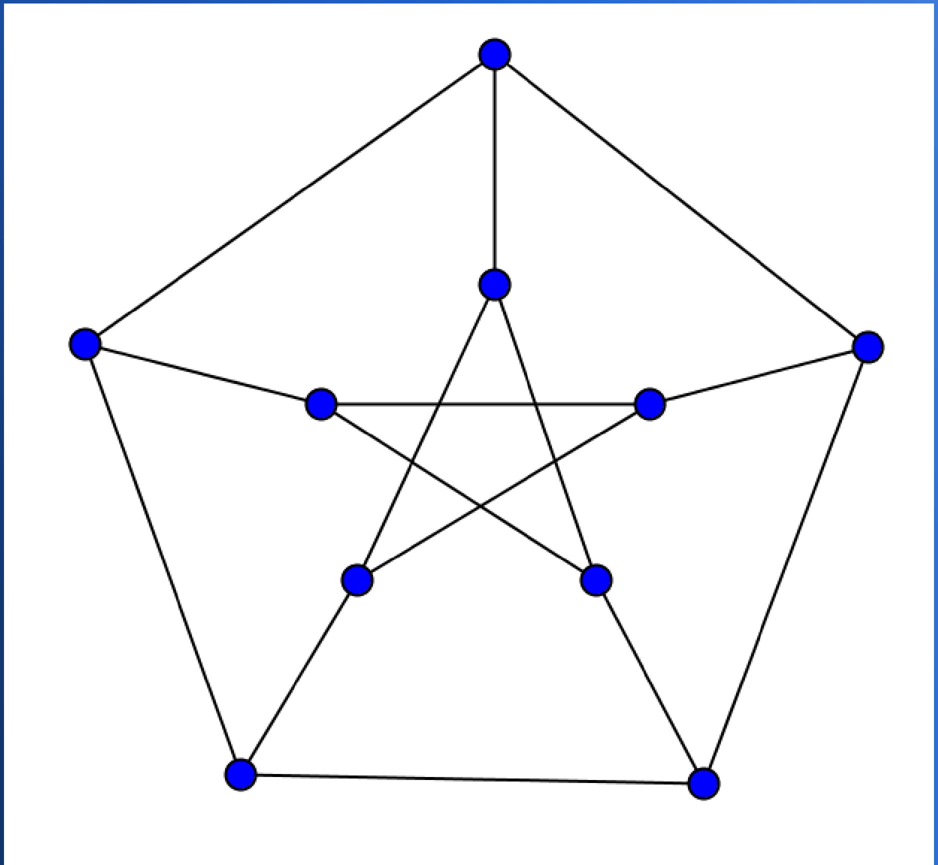
Vertex Stable

A graph G is called *(H;k)-vertex stable* or *(H;k)-stable* if G contains a subgraph isomorphic to H even after removing any k of its vertices.

Example: Consider the Petersen Graph and we're going to check its stability with $H=C_5$.

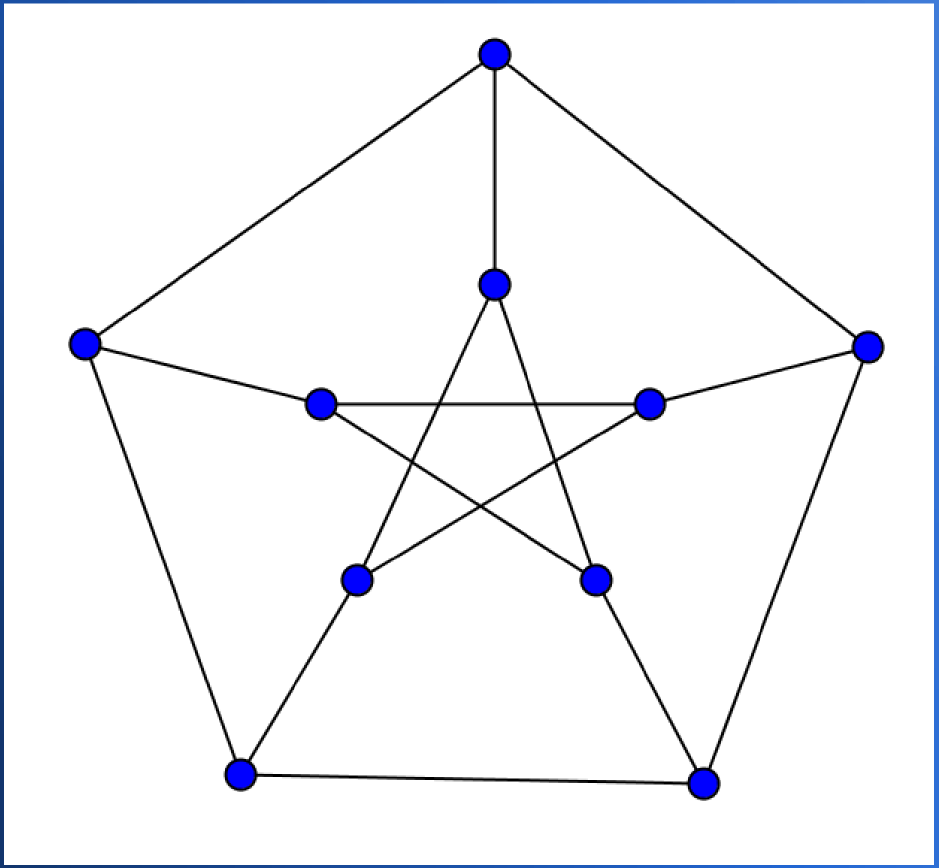
Example

Is the Petersen Graph (C_5;0)-stable?



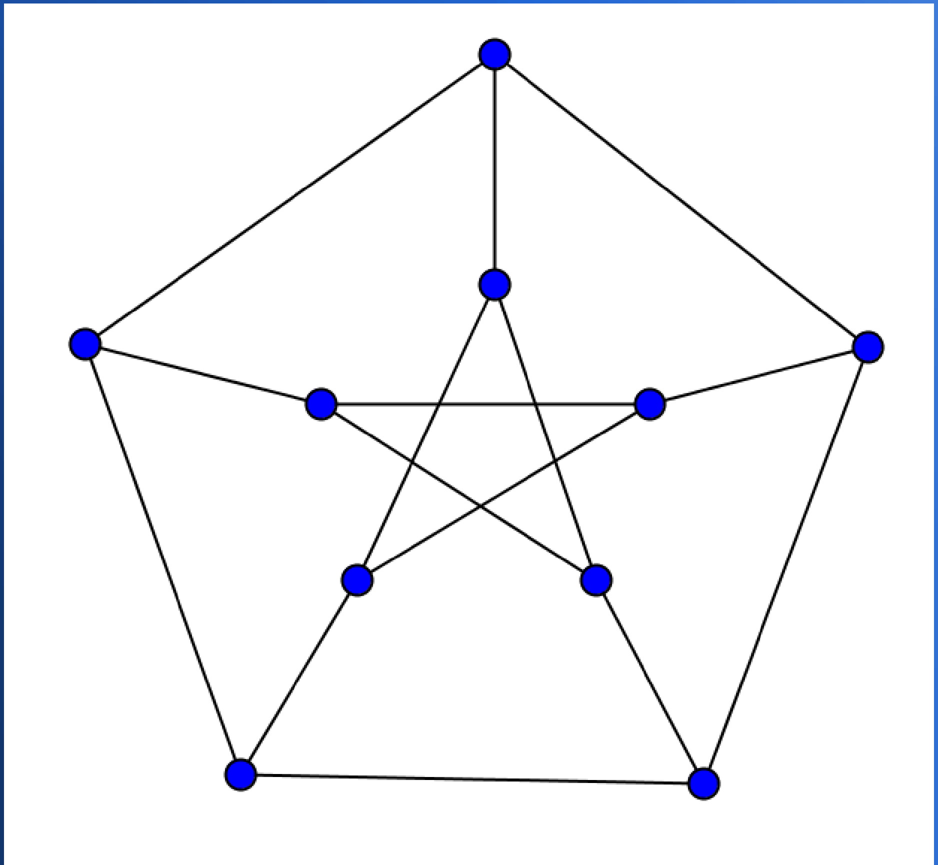
Example

Is the Petersen Graph (C_5;*1*)-stable?

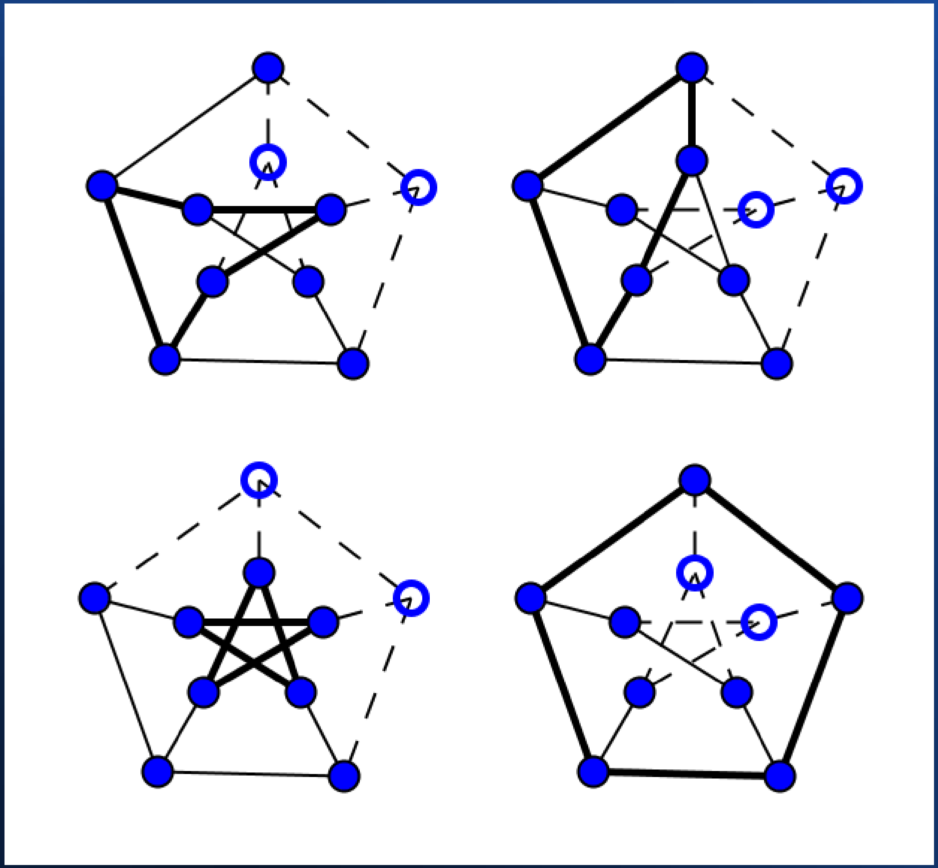


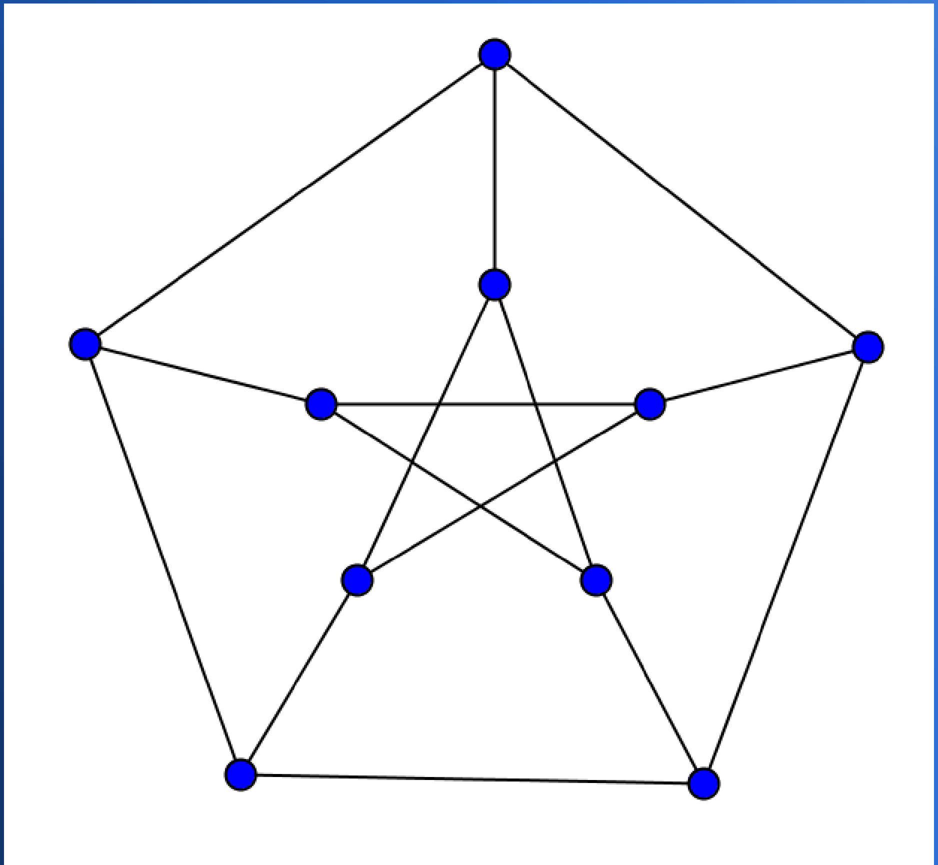
Example

Is the Petersen Graph (C_5;2)-stable?



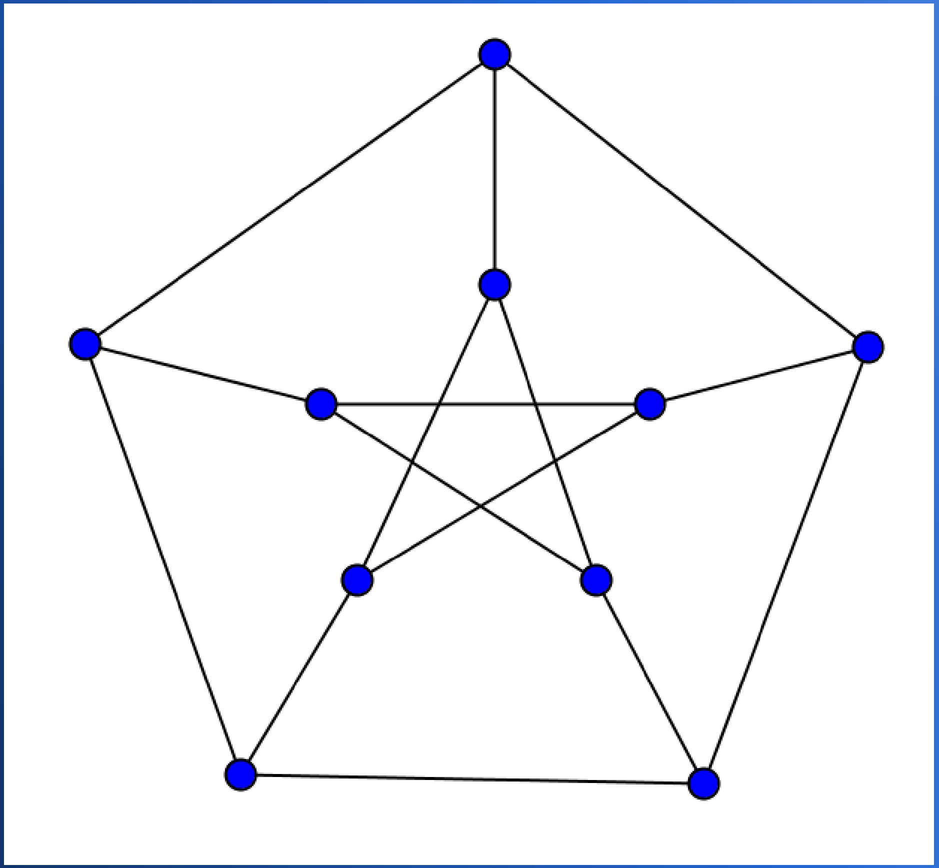
Example



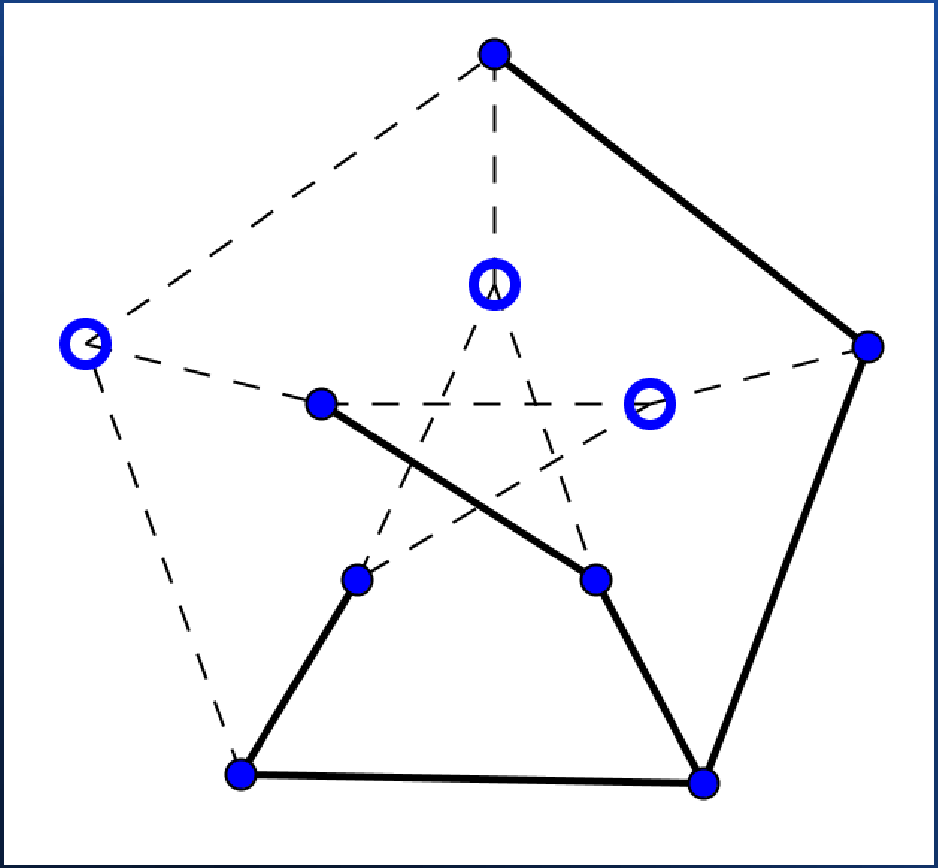


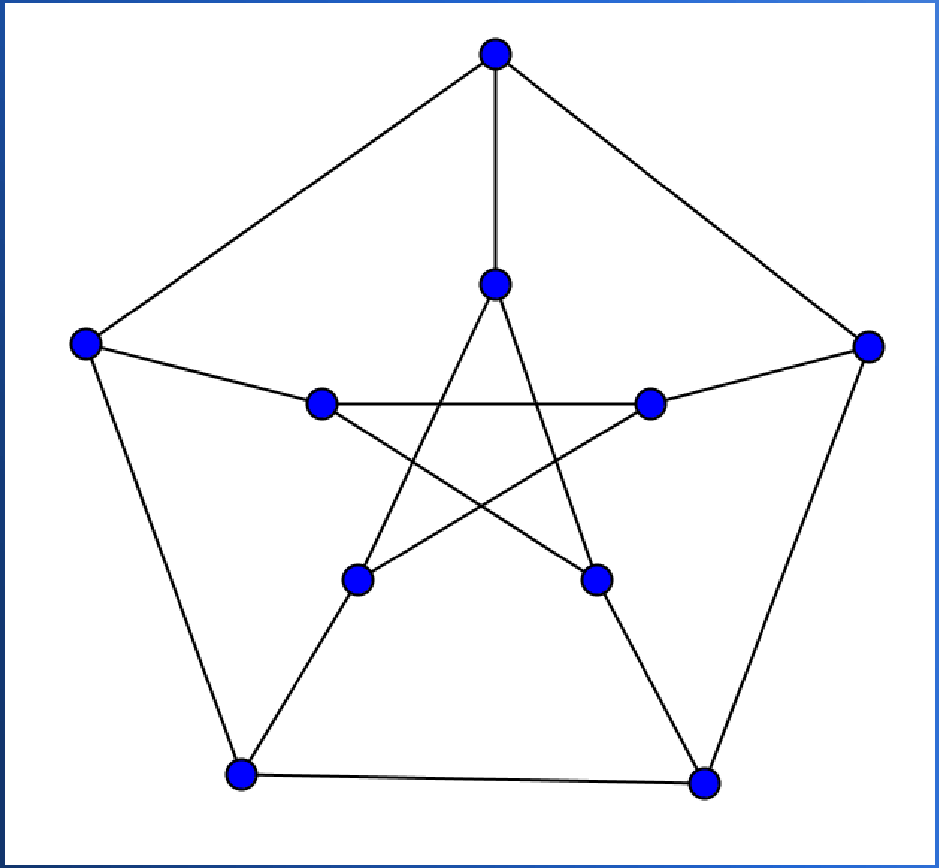
Example

Is the Petersen Graph $(C_5;3)$-stable?



Example





minimum (H;k)-stable graph

If G has the minimum size of any (*H;k*)-stable graph, then

$$\|G\| := stab(H;k),$$

and we refer to G as a minimum (*H;k*)-stable graph.

Further assumptions

We will not consider isolated vertices in any of the graphs in question.

Why? After adding to or removing from an ($H;k$)-stable graph any number of isolated vertices, we still have an ($H;k$)-stable graph with the same size.

General Bounds

The goal is to find a lower bound on the size of our stable graph, G, for selected graph H and arbitrarily chosen k.

Major problem to address:

We need to be able to remove *any* set of k vertices without losing a subgraph of H.

Theorem

Let δ_H be the minimum degree of H. If G is a minimum (*H;k*)-stable graph, then

$$|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geqslant k+1.$$

Moreover, if G is not a union of cliques, then the inequality is strict.

(This is the light at the end of the tunnel.)

Lemma

If G is a minimum (*H;k*)-stable graph, then every vertex and every edge of G belongs to some subgraph of G isomorphic to H.

Proof: Roughly, imagine that you have a minimum (*H;k*)-stable graph where there is a vertex or edge that doesn't belong to a subgraph of G isomorphic to H. If that edge or vertex is never used in any of the isomorphisms, then its removal doesn't affect the stability of G. Hence G was not minimum! ■

Proposition

Let δ_H be the minimum degree of a graph H. Then in any minimum (*H;k*)-stable graph G,

$$d_G(v) \geq \delta_H$$

for each vertex $v \in V(G)$.

Proof: Follows from the previous lemma.

The Ordering...

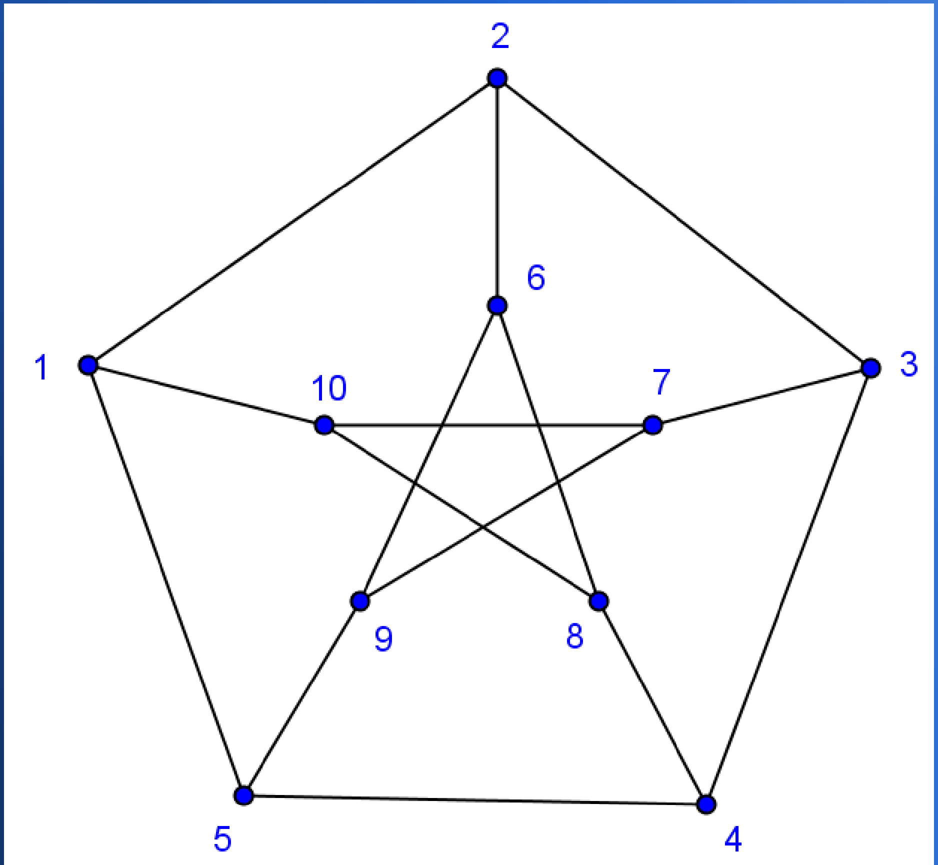
Fix any ordering σ of V(G). Let $deg_{\sigma}(v)$ denote the number of neighbors of that are on the left of v in ordering σ. Then let S_{σ} denote the set of all vertices with $deg_{\sigma}(v) \leq \delta_H - 1.$

If remove from G the set of vertices in $V(G) \setminus S_{\sigma}$, the claim is that this will induce a subgraph on G that will contain no copies of H.

What?

Example

Consider a labeling of our previous graph, the Petersen Graph. Further, suppose that H is again C_5.



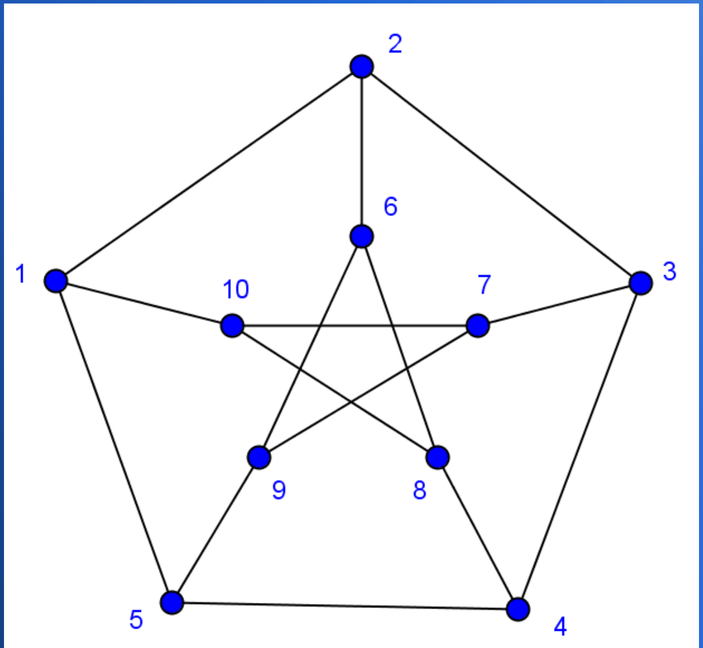
Example

Observe that

$$S_\sigma := \{ v_i | \, deg_\sigma(v_i) \leq 1 \}$$
$$= \{ v_1, v_2, v_3, v_4, v_6, v_7 \}$$

Hence

$$V(G) \backslash S_\sigma := \{ v_5, v_8, v_9, v_{10} \}$$



n	1	2	3	4	5	6	7	8	9	10
$deg_\sigma(v_n)$	0	1	1	1	2	1	1	2	3	3

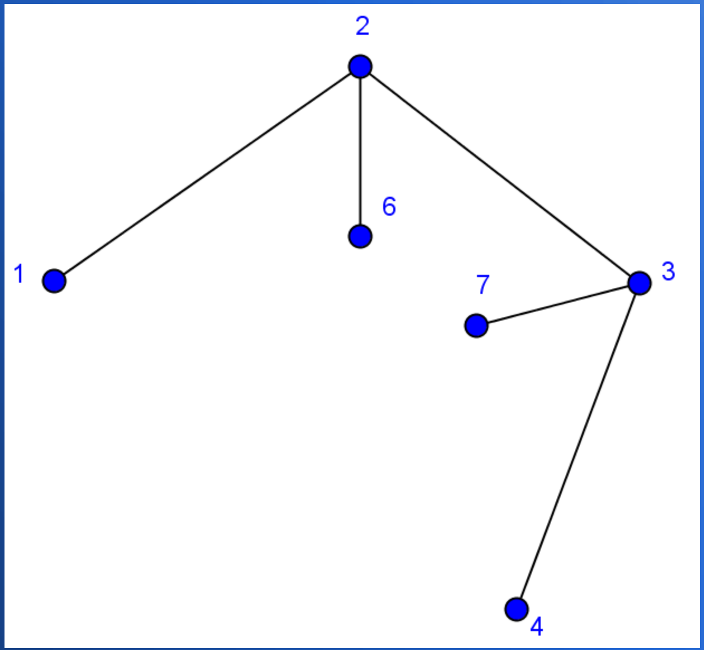
Example

Observe that

$$S_\sigma := \{v_i \mid deg_\sigma(v_i) \leq 1\}$$
$$= \{v_1, v_2, v_3, v_4, v_6, v_7\}$$

Hence

$$V(G) \setminus S_\sigma := \{v_5, v_8, v_9, v_{10}\}$$



n	1	2	3	4	5	6	7	8	9	10
$deg_\sigma(v_n)$	0	1	1	1	2	1	1	2	3	3

So how does it work?

In general, with any ordering we destroy all copies of H by consecutively eliminating all vertices of S_σ. Each vertex in S_σ has left degree $\leq \delta_H - 1$ and thus cannot be the rightmost vertex of any copy of H in the induced subgraph.

From this we get the following:

$$|G| - |S_\sigma| \geq k + 1$$

What next?

We need to find a way to approximate the size of S_σ. To do this we will need...

Probability and Expected Values.

Lets count some stuff!

The story.

Our story begins with letting σ be an ordering on a vertex set of size n... sitting somewhere in this ordering is an arbitrary vertex v, with degree equal to $d_G(v)$. Also sitting within this ordering is all of v's neighbors.

$$\_\_ \cdots \_\_ \cdots \_\_ \cdots \_\_ \cdots \_\_ \cdots \_\_ \cdots \_\_$$

$$\_\_ \cdots \_\_ \cdots \_\_ \cdots \underline{v} \cdots \_\_ \cdots \_\_ \cdots \_\_$$

$$\underline{v_1} \cdots \underline{v_2} \cdots \underline{v_3} \cdots \underline{v} \cdots \underline{v_4} \cdots \underline{v_5} \cdots \underline{v_6}$$

Choosing spots

The vertex v has $d_G(v)$ neighbors for which each gets a spot in the ordering. Don't forget v needs a spot too! So we get the following:

$$\binom{n}{d_G(v)+1}$$

Choosing spots

Recall that $deg_{\sigma}(v) \leq \delta_H - 1$. We have that if we assign v to any of these first δ_H spots, we satisfy $deg_{\sigma}(v) \leq \delta_H - 1$. Hence we get:

$$\binom{n}{d_G(v)+1}(\delta_H)$$

The rest of the counting is assigning the remaining vertices to their spots and dividing by the total number of permutations on n vertices.

The Probability

$$Pr(deg_\sigma(v) < \delta_H) = \frac{\binom{n}{d_G(v)+1}(\delta_H)(d_G(v))!(n-d_G(v)-1)!}{n!}$$

$$= \frac{\delta_H}{d_G(v)+1}$$

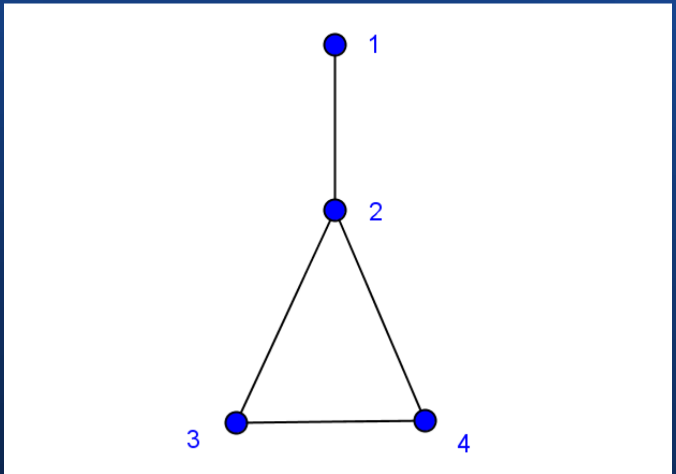
The Expectation

By using a method of expected values we get...

$$Pr(v \in S_\sigma) = \frac{\delta_H}{d_G(v)+1}$$

$$E(|S_\sigma|) = \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}$$

Example



1234	2134	3124	4123
1243	2143	3142	4132
1342	2314	3214	4213
1324	2341	3241	4231
1432	2413	3412	4312
1423	2431	3421	4321

Set $\delta_H = 2.$

$$E(|S_\sigma|) = \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1} = \frac{2}{1+1} + \frac{2}{3+1} + \frac{2}{2+1} + \frac{2}{2+1}$$

$$= \frac{24}{24} + \frac{12}{24} + \frac{16}{24} + \frac{16}{24} = \frac{68}{24} \approx 2.8\overline{33}$$

Thus...

$$E\left(\left|S_\sigma\right|\right)=\sum_{v\in V(G)}\frac{\delta_H}{d_G(v)+1}$$

Gives us that

$$\left|G\right|-\left|S_\sigma\right|\geq k+1$$

$$\left|G\right|-\sum_{v\in V(G)}\frac{\delta_H}{d_G(v)+1}\geq k+1$$

■

Corollary

Let H be any graph and let δ_H denote the minimum degree of H. Then

$$stab(H;k) \geq (k+1)\left(\delta_H + \sqrt{\delta_H(\delta_H - 1)} - \frac{1}{2}\right).$$

What?

Corollary

Roughly, with the use of...

Lemma: The expression

$$\sum_{k=1}^{l} \frac{1}{x_k} \quad \text{with} \quad \sum_{k=1}^{l} x_k = r\,, r \in \Re$$

is minimal if all the x_j are equal. (Uses constrained optimization.)

Example: Set $r=30$,

$$30 = 9+8+7+2+2+2 \qquad \frac{1}{9}+\frac{1}{8}+\frac{1}{7}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{947}{504}\approx 1.879$$

$$30 = 5+5+5+5+4+6 \qquad \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{4}+\frac{1}{6}=\frac{73}{60}\approx 1.2167$$

$$30 = 5+5+5+5+5+5 \qquad \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{6}{5}=1.2$$

Corollary

Using this fact, note that if the average degree of G is defined:

$d_G = \frac{2\|G\|}{|G|}$ we get: $\|G\| = \frac{d_G |G|}{2}$. Then

$$\sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq \sum_{v \in V(G)} \frac{1}{d_G+1} = \frac{|G|}{d_G+1}.$$

Rearranging the results from our first theorem we get

$$|G| \geq k+1+\sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1} \geq k+1+\frac{|G|}{d_G+1}.$$

Putting all of this together we and applying a bunch of algebra...

$$|G| \geq (k+1)\frac{d_G+1}{d_G+1-\delta_H}.$$

Corollary

...and using the quadratic formula on a characteristic polynomial and then applying the first derivative test to verify the minimum point. It follows that...

$$\|G\|=\left(\frac{d_G}{2}\right)|G|\geq\frac{k+1}{2}\cdot\frac{d_G(d_G+1)}{d_G+1-\delta_H}\geq(k+1)\left(\delta_H+\sqrt{\delta_H(\delta_H-1)}-\frac{1}{2}\right).$$

Since G was assumed to be minimal we get that...

$$stab(H;k)\geq(k+1)\left(\delta_H+\sqrt{\delta_H(\delta_H-1)}-\frac{1}{2}\right).$$

Complete Graphs, K_q

Nice thing about complete graphs: All the vertices have the same degree... so if we set H to be K_q, then $\delta_H = q - 1$.

Couple this fact with our previous theorem, corollary, and TONS of algebra, we get...

Theorem

Let G be a $(K_q;k)$-stable graph, $q \geq 2$ and $k \geq 0$ then

$$\|G\| \geq (2q-3)(k+1)$$

with equality if and only if G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

The equality is simply a nice consequence from selecting cliques of consecutive sizes. Simple algebra proves this... A lot of simple algebra.

The coin problem...

is a mathematical problem that asks for the largest monetary amount that cannot be obtained using only 2 coins of specified denominations (relatively prime).

In other words, there should exist a positive integer such that for any integer greater than it, there exists a linear combination of the two denominations to express the value.

Frobenius Numbers

The largest nonnegative integer that can't be expressed as a linear combination of the two relatively prime denominations is called the Frobenius Number.

Idea by Georg Frobenius.

The following by James Joseph Sylvester:

Given positive integers m,n where $gcd(m,n)=1$, the Frobenius number $m(a)+n(b)=K$ is given by $mn-m-n.$

Georg Frobenius

Example

$$4(1)+5(0)=4$$
$$4(0)+5(1)=5$$
$$4(2)+5(0)=8$$
$$4(1)+5(1)=9$$
$$4(0)+5(2)=10$$
$$4(3)+5(0)=12$$
$$4(2)+5(1)=13$$
$$4(1)+5(2)=14$$
$$4(0)+5(3)=15$$
$$\vdots$$

Confirming with the closed formula:

$$m=4, n=5$$
$$mn-m-n$$
$$4\cdot5-4-5=11$$

Frobenius number for 4,5 is 11!

McNugget Problem is a “famous” example of the use for Frobenius Numbers.

Theorem

Let $q \geq 2,\ k \geq 0$ be non-negative integers. Then

$$stab(K_q; k) \geq (2q-3)(k-1),$$

with equality if and only if $k = a(q-2) + b(q-1) - 1$ for some non-negative integers a, b. In particular,

$$stab(K_q; k) = (2q-3)(k-1), \quad for \quad k \geq (q-3)(q-2) - 1.$$

Furthermore, if G is $(K_q; k)$-stable graph with

$$\|G\| = (2q-3)(k+1)$$

then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

Future Work

- Things to think about:
  - Uniqueness of minimal stable graph?
  - Families of graphs that guarantee certain stability. For example...
- Can prove:
  - C_{kn+k+1} is $(P_n;k)$-stable.
- Conjecture (Fermat-style):
  - C_{kn+k+1} is the unique minimal $(P_n;k)$-stable graph.
  - $K_{q,r,s}$ is $(C_{2n+1};k)$-stable.
    - $K_{q,r}$ is $(C_{2n};k)$-stable.

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The End

Thank you!

Special Thanks:

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Mark and Janet